Differential techniques

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1 Background

Physicists love differentials.

There is a reason for this. Of all the powerful tricks that physicists have devised to find a complete description of the inner workings of the Universe, perhaps none has been more important and useful than the following:

> When trying to understand a large, complicated system, just consider what happens locally, at a single point.

One can think of this technique as the physics anaology to the classical mathematical technique of *induction*.

But in physics, our tooling looks a little bit different. Since we primarily work in continuous space, we are made to work with differentials, instead of integer steps.

This technique which I will showcase below, along with a couple of other ways of manipulating differentials, is the subject of this handout. The generality of this technique can be seen by the diversity of physical problems considered in this handout[1] Once you have mastered this technique, you will have added a very powerful tool in your toolbox as a problem-solving physicist.

Example: Atmospheric pressure

You may have heard that pressure is lower further up in the atmosphere. Let's prove that this follows naturally from a simple model!

Let's consider the atmosphere as having a constant temperature T.[2] Then, in accordance with our technique, we consider a small parcel of air of area A and an infinitesimal (really really small) height dh. This means that the parcel has an infinitesimal volume

$$dV = A \times dh$$

Now, what we're going to do is we're going to considering the force balance on this parcel. For the atmospheric parcel to be in equilibrium, the forces on the parcel must be balanced (by Newton's first law). The two forces are *gravitational* (downward) and *pressure* (which, to balance, must be net upward). Let's write them down:

$$dF_G = -g \times dm$$
$$dF_P = A \times (P(h) - P(h + dh))$$

We want our differentials to all just be in terms of one variable, so we want to express and substitute dm in terms of dh. So we write:

$$dm = \rho \times dV = \rho A \times dh$$

and from the internal gas law $\rho = P\mu/RT$

[1] This diversity also means that many questions will make use of different physical principles and formulas, which should be at the high school physics level (and where not, I have tried to include them as hints).

[2] This is in fact not a good approximation, but will serve our purposes for introducing the differential technique.

So,

$$dF_G = -g\frac{P\mu}{RT}A \times dh$$

Now, we're ready to balance forces:

$$dF_G + dF_P = 0$$
$$-g\frac{P\mu}{RT}A \times dh + A \times (P(h) - P(h + dh)) = 0$$
$$\frac{P(h + dh) - P(h)}{dh} = -g\frac{P\mu}{RT}$$
$$\frac{dP}{dh} = -g\frac{P\mu}{RT}$$

A differential equation describing our atmosphere! This you can solve, and find the atmospheric pressure at some given height h. Try it!

Question: Solve the differential equation

2 Questions

1. The surface area of a sphere of raidus r is given by $A = 4\pi r^2$. By summing together the infinitesimal area alements of many spherical shells, show that the volume of a ball is given by $V = 4\pi r^3/3$.

2. The moment of inertia I of a body around an axis of rotation is defined as $I = \int_V r^2 dm$, there V is the volume of the body, r is the shortest distinace between a given point on (or inside) the body and the axis of rotation, and dm is the inifitesimal mass of that point on the body. By rewriting the infinitesimal mass dm in termfs of ρ and dr, where ρ is the density of the body, evalate the moments of inertia of the following bodies:

   (a) A solid sphere of radius R and mass M, rotating around its own axis.

   (b) An spherical shell of radius R and mass M, rotating around its own axis.

   (c) A long, thin rod of length L and mass M, rotating around an axis thorugh its center of mass.

   (d) A long, thin rod of length L and mass M, rotating around an axis thorugh one of its far edges.

3. The acceleration of a boat depends on its speed as shown in figure 1. The boat is given initial speed $v_0 = 4$ m/s. What is the total distance travelled until the boat will almost come to rest? (Kalda, Kinematics)

4. A uniform chain of total length L and mass M hangs vertically from a fixed point. Find the tension $T(x)$ in the chain as a function of distance x measured downward from the top.

5. A meteor is heading towards Earth. To stop the meteor, it is decided that it should be blown up into pieces. The meteor is perfectly round, has a radius of 1 km and a density of 103 kg/m^3. When the meteor is to be blown up, it is desirable that all the particles escape the meteor's gravitational field so that no parts of the meteor fall back against each other and collect. How much energy will this require at least? Assume that the pieces of the meteor should be very small. (Norwegian Physics Olympiad, 2021)

6. **The rocket equation** A rocket of initial mass M_0 (including fuel) burns fuel at a constant rate and ejects exhaust at velocity v_e relative to the rocket. The rocket starts from rest in deep space (no gravity or air resistance).

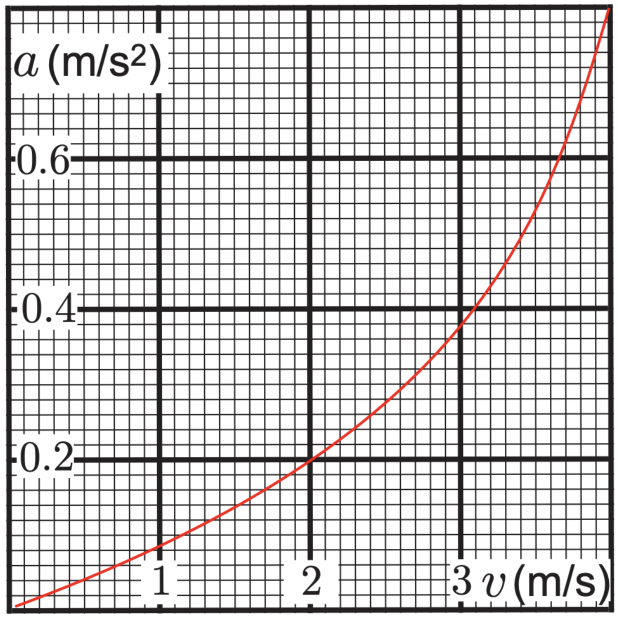

Figure 1: Plot of a vs v

(a) Consider the rocket at time t when its mass is $M(t)$ and velocity is $v(t)$. In a small time interval dt, the rocket ejects mass dm of exhaust. Using conservation of momentum for the system (rocket + ejected exhaust), show that the equation of motion of the system is given by:

$$M\frac{dv}{dt} = -v_e\frac{dM}{dt}$$

Hint: Consider the momentum of the system before and after ejecting the exhaust mass dm.

(b) Integrate the differential equation to find the rocket's velocity v in terms of its initial mass M_0, current mass M, and exhaust velocity v_e. When all fuel is consumed, the rocket has final mass M_f. What is the minimum exhaust velocity needed to achieve orbital velocity (≈ 8 km/s) if the fuel constitutes 90% of the initial rocket mass?

7. A spherically symmetric body of total mass M and radius R has constant density ρ throughout and is in hydrostatic equilibrium under its own gravity.

(a) Set up the hydrostatic equilibrium condition by considering the balance of forces on a thin spherical shell at radius r. Show that this leads to the differential equation:

$$\frac{dP}{dr} = \frac{GM(r)\rho}{r^2}$$

where $M(r)$ is the mass within radius r. Then express $M(r)$ in terms of the constant density ρ and derive the differential equation for $P(r)$ in terms of ρ, G, and r only.

(b) Solve the differential equation by direct integration. Apply the boundary condition $P(R) = 0$ to find the complete pressure profile $P(r)$.

(c) Express your final answer in terms of M and R (eliminate ρ), and find the central pressure $P(0)$. For Earth ($M \approx 6 \times 10^{24}$ kg, $R \approx 6.4 \times 10^6$ m), estimate this central pressure.

8. A semitransparent disk (see figure 2) with radius $a = 15.0$ mm, thickness $b = 0.2$ mm and is composed of a material having thermal conductivity $k = 0.3\ \mathrm{W\,m^{-1}\,K^{-1}}$. The outer rim of the disk is thermally connected to a circular metallic holder (not shown in the figure) maintained at a constant temperature

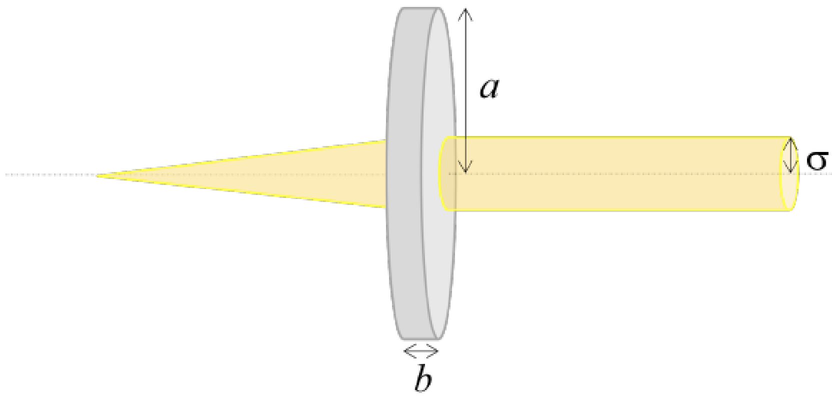

Figure 2: Thermal lens

$T_h = 20°$ C. A parallel laser beam of radius $\sigma = 0.5$ mm and power $P_L = 20$ mW is incident normally onto the center of the disk. The intensity distribution is homogeneous across the cross-section of the beam.

You may find it useful to know that the heat flux (flow of heat per unit area per unit time) Q between two points of temperature difference ΔT separated by a distance Δx is given by $Q = k\frac{\Delta T}{\Delta x}$ where k is the thermal conductivity.

(European Physics Olympiad, 2023)

(a) Sketch a qualitative graph of the temperature profile $T(r)$, where r denotes the distance from the axis of the beam. Indicate clearly on the graph the illuminated region $0 \leq r \leq \sigma$ and the outer region $\sigma \leq r \leq a$.

(b) In the vicinity of the center of the disk, the temperature profile can be represented by a quadratic function $T(r) = T_c + mr^2$. Calculate the parameters T_c and m.

9. **The age-old catenary problem** A chain is suspended from two points on the ceiling a distance d apart. The chain has a uniform mass density λ, and cannot stretch. Find the shape of the chain.